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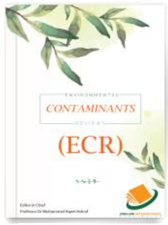

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RESEARCH ARTICLE

AN ANALYSIS OF TWO-DIMENSIONAL FLOW THROUGH A WATER RESERVOIR USING MATHEMATICAL APPROACH

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ABSTRACT

Optimal Variational Iteration Method (OVIM) is Variational Iteration Method (VIM) coupled with auxiliary parameter *h*. In this paper, we have discussed a hydrological problem with pressure distribution phenomenon solved with Optimal Variational Iteration Method (OVIM). In the framework of Optimal Variational Iteration Method (OVIM), the auxiliary parameter *h*, that is convergence controlling parameter is the primary tool which guarantees the convergence of said technique. Moreover, the convergence is obtained by so-called residual error method. Results shows that the reliability of the method with the least error and provide the required solution to the pressure distribution of water in a water reservoir in initial four iteration.

KEYWORDS

Optimal Variational Iteration Method, hydrological problem, Variational Iteration Method.

1. INTRODUCTION

Differential equations exhibit an extraordinary aptitude to say what will take place in future in the world. They appear in almost every field, sciences or arts. They can picture of almost every phenomenon of real life. In fact, differential equations can explain our whole life in a systematic way. Mainly, differential equations are of two types, Linear and nonlinear one. Most of the phenomenon of sciences and engineering are modeled by nonlinear ordinary and many times partial differential equations. Seek an accurate, better and natural approach for the solution is escalating day by day. Nonlinear differential equations are generally very complicated and cannot be resolved accurately. In some cases, even if the exact solution is, let say, practicable, there are too many calculations involved, making resulting solution impossible to achieve or challenging to construe.

Several analytical methods are there proposed by many types of research for finding the approximate solution. For example, Method of Adomian Decomposition, Method of Homotopy Perturbation, Method of Homotopy Analysis, etc [1-4]. One of the techniques that have turn out to be very popular these days is the Variational Iteration Method. This technique, initially, was given and idea of the said technique [5,6]. The method initiates a consistent and useful methodology for an ample assortment of scientific and engineering applications. The variational iteration method applies to numerous nonlinear problems of diverse nature. In this technique, we need a very few numbers of iterations for a reliable approximate solution. Much research has been done, and many interesting problems are done by the said method; given in [7-13].

Hydrology is a branch of science dealing specifically with the flow of water. The same subject is dealt with in Fluid mechanics in mathematics [14-20]. Much work has been done on fluid flow in mathematics [18-19]. Fluid flow is not a simple quantity, and It is further dependent on many quantities for example influx and outflux rate, pressure distribution, gravity at the point of flow, and most important, viscosity of fluid. Let us consider an underground flow between two walls. For simplicity, we suppose that the adjacent layers of fluid are independent of effect of transmissibility of each other [20]

$$\frac{\partial J}{\partial t} = \nabla \cdot (\mathrm{L(y)} \nabla J) + G(\mathrm{y,t}) \qquad (1)$$

Equation (1) is a mathematical model of a hydrological equation which is generated by the flow of water [20] through the reservoir coupled with equation (2).

$$div(\bar{\bar{R}}\, gradB) + W = \upsilon \frac{\partial B}{\partial t} \qquad (2)$$

where $G(\mathrm{y,t})$ is Influx or outflux rate of fluid, $J(\mathrm{y,t})$ is Pressure distribution of fluid, $\mathrm{L(y)}$ is Transmissibility of reservoir, υ is storing coefficient and B is observation from piezometric level. To understand the framework of Optimal Variational Iteration Method (OVIM), one should start with the methodology of Variational Iteration Method (VIM) from its very initial stage.

2. METHODOLOGY

The Variational Iteration Method (VIM) is a concept of J.H. He who published its basic algorithm and application in 1998 and developed the method fully in 2006. The said method has been used by numerous mathematicians in order to solve a diverse range of linear, nonlinear, parametric, parabolic, integral, and fractional kind of ODEs and PDEs. This method is famous for a lesser number of iterations needed for an accurate and rapid convergent solution. The term optimal refers to a particular mechanism called optimization. Optimization deals with a problem for a point at which the required function is minimized or maximized. Generally, that point has to satisfy one or more constraints. Optimization is a very vast field and further divided into several branches which depend on the type of objective function and nature of constraints. Linear programming concerns the situation when both of them are linear. Simplex famous is famous for this purpose.

The basic concept of Optimal Variational Iteration Method (OVIM) is hidden in the correctional function of required problem, that is [15]:

$$u_{n+1}(t) = u_n(t) + h\int_0^t \lambda\{Lu_n(s) + N\tilde{u}_n(s)\}\,ds \tag{3}$$

Here λ is characterized as the Lagrange Multiplier and its value can be calculated from variational theory, u_n is the nth term of a solution in series, L Is linear operator for the linear part of the equation and N is nonlinear operator for nonlinear terms of equation.
To apply Optimal Variational Iteration Method on Eq (1), we transform it in the following form [20]

$$\frac{\partial J}{\partial t} = \frac{\partial^2 J}{\partial y^2} - e^{-t} + \pi^2 \sin(\pi y), \quad y \in [0,1] \tag{4}$$

The eq (4) is a partial differential equation exhibiting two-dimensional flow of fluid between two walls in a reservoir. Here the influx or outflux rate is $-e^{-t} + \pi^2 \sin(\pi y)$ which is denoted by $G(y,t)$ in eq (1). Initial and boundary conditions associated with eq (4) are [20]:

$$J(0,t) = J(1,t) = e^{-t} \qquad 0 \le t \le T \tag{5}$$

$$J(y,0) = 1 + \sin(\pi y) \qquad 0 \le y \le 1 \tag{6}$$

The correctional functional of Eq (4) reads,

$$J_{n+1}(t) = J_n(t) + h\int_0^t \left\{\frac{\partial^2 J}{\partial y^2} - \frac{\partial J}{\partial t} - e^{-t} + \pi^2 \sin(\pi y)\right\} dt \tag{7}$$

We start with initial guess taken as $J_0 = 1 + \sin(\pi y)$. Taylor series of terms $-e^{-t} + \pi^2 \sin(\pi y)$ is calculated as

$$TaylorApproximation\left(-\exp(-y) + \pi^2 \cdot \sin(\pi x), [x, y] = [1, 0], 5\right) =$$
$$\frac{1}{120}y^5 - \frac{1}{24}y^4 + \frac{1}{6}y^3 - \frac{1}{2}y^2 + y - 1 + \frac{1}{6}\pi^5 (x-1)^3 - \pi^3 (x-1) - \frac{1}{120}\pi^7 (x-1)^5$$

And then eq (7) can be written as:

$$J_{n+1}(t) = 1 + \sin(\pi y) + h\int_0^t \left\{\frac{\partial^2 J}{\partial y^2} - \frac{\partial J}{\partial t} + (\text{Taylorapproximation})\right\} dt \tag{8}$$

Here *h* is a constantly called convergence controlling parameter. The presence of *h* plays a very pivotal part which cannot be managed by Lagrange multiplier used in that of classical VIM. It is obvious when parameter *h* becomes equal to one; the present equation will be converted into classical Variational iteration method. The approach of convergence controlling parameter gives a considerable extensible way of attaining the successive iteration with the Variational iteration method when *h* is coupled. From an analytical approach, the selection of *h* can be considered not to affect our solution. The region of validation of parameter h by drawing constant *h-curves* for some fixed values of our solution. It is possible for every real-life problem by taking out some fixed values of that solution which is not equal to zero and then plot it against *h* and try to obtain the interval of *h* where some minor change in values is seen. A better approximation of convergence controlling parameter *h* is obtained by residual error method in which we impose an assumption that we want to find the solution in a specific interval. Finally, minimizing the residual error equation gives the value of *h*.

Using residual error method, we identify h=*0.599*. The first four iterations are given as

$$u_1 = -1.00000000010^{-11} t \begin{pmatrix} 5.91189303610^{11} \sin(3.141592654x) - 9.98333333310^{9} y^3 \\ +2.99500000010^{10} y^2 \end{pmatrix} \tag{9}$$

$$u_2 = 1.00000000010^{-8} t \begin{pmatrix} 1.74752396410^{9} \sin(3.141592654x)t - 8.97002510^{6} y^2 t + \\ 1.794005010^{7} yt + 5.990000010^{7} y - 5.990000010^{7} \end{pmatrix} \tag{10}$$

$$u_3 = -1.00000000010^{-11} t \begin{pmatrix} -3.58202998310^{9} t^2 y + 3.58202998310^{9} t^2 + 3.44372492010^{12} t^2 \\ \sin(3.141592654x) + 1.79400500010^{10} t - 7.30801358710^{12} x + \\ 1.19782054710^{12} - 3.05509652010^{12} x^3 + 9.16528956010^{12} x^2 \end{pmatrix} \tag{11}$$

$$u_4 := 1.00000000010^{-10} t \begin{pmatrix} 5.08973334510^{11} t^3 \sin(3.141592654x) - 5.364089910^{7} t^3 \\ +5.49000844510^{11} tx - 5.49000844510^{11} t - 1.50762970310^{11} x^5 \\ +7.53814851510^{11} x^4 - 1.50762970310^{12} x^3 + 1.50762970310^{12} x^2 \\ -7.53814851510^{11} x + 1.50762970310^{11} \end{pmatrix} \tag{12}$$

The exact solution of Eq (8) is $e^{-t} + \sin(\pi y)$. After four iterations, the graphical representation of the numerical result calculated from Variational Iteration Method (VIM) is compared with the exact solution.

3. RESULTS AND DISCUSSION

In present work, the basic idea and working of Optimal Variational Iteration Method (OVIM) have been discussed in detail. Consequently, the implication of said technique on a partial differential equation shown by equation (8) exhibiting a water flow phenomenon is examined and the results are quite astonishing. From equation (9) to equation (12), there is a gradual increase in the fidelity of solution which eventually results in an approximate solution approaching the exact solution [21]. The error analysis shows that the error between the exact solution and first iteration of Optimal Variational Iteration Method (OVIM) is 0.025 which has been calculated through Maple using $Error[i] := E[i] - A[i]$.

Where, $E[i] = exact$ and $A[i] = approminate$ for each iteration i. The value of error decreases with each iteration and finally reaches approximately zero after a number of iterations. Graphical representation in two different domains in Figure 1 and Figure 2 is showing the effectiveness of method. The residual error method is used for a reliable calculation. Graphical representation of results is obtained by Maple coding shows the 3d image of pressure distribution from x= -1 to x=1 and t=0 to t=0.1. It was found that a sudden fall in pressure distribution in the start and end region, in the middle area, the pressure distribution remains almost constant. Figure 2 shows the 3d image of pressure distribution from x=0 to x=1 and t=0 to t=1. Here reported that a fall in the value of pressure distribution of water when both x and t approaches to 0.

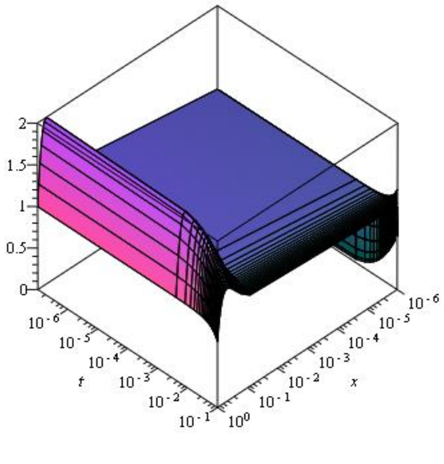

Figure 1: 3d image of pressure distribution from x= -1 to 1 and t= 0 to 0.1

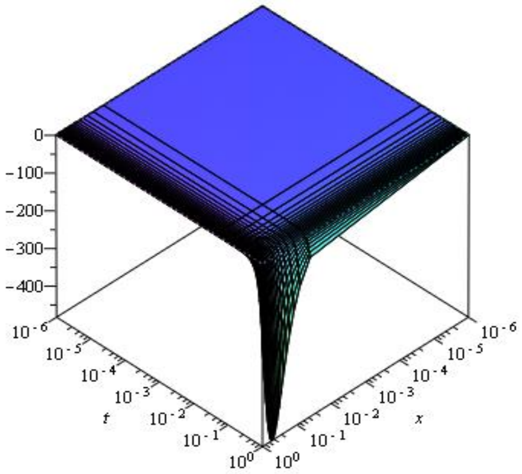

Figure 2: 3d image of pressure distribution from x= 0 to 1 and t= 0 to 1

4. CONCLUSION

Partial differential equation representing a two-dimensional flow in a reservoir between two walls having a flux rate equal to $-e^{-t}+\pi^2\sin(\pi\,y)$. Since the flux here is a nonlinear term, the Taylor series is replaced for better calculation. The analysis showed that the optimal variational iteration method is beneficial for solving nonlinear differential equations among some iterative methods. A few numbers of iteration are enough to obtain very highly accurate solution and approximated solution is found much closer to exact solution.

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