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RESEARCH ARTICLE

A STUDY OF HYDROLOGICAL BEHAVIOR OF WATER-BEARING USING HOMOTOPY PERTURBATION METHOD

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ABSTRACT

This paper provides an overview of the usage of mathematical models in hydrology. A partial differential equation is solved with the help of Homotopy Perturbation Method (HAM). The equation is taken from flow of fluid in a reservoir. Results obtained shows that HAM is an effective method for these types of equations and provides a convergent solution.

KEYWORDS

Homotopy Perturbation Method (HAM), the flow of fluid, convergent solution.

1. INTRODUCTION

Mathematics is the language of other sciences and Hydrology is a branch of science dealing exactly with the flow of water [1]. Differential equation exhibits different physical phenomena and processes in most of the industries. These equations are a mathematical model of that particular problem. For example

$$\frac{\partial u}{\partial t} = \nabla \cdot (k(x)\nabla u) + \mathrm{f}(x,\mathrm{t}) \quad (1)$$

In this case, the equation exposes the hydrological characteristic of water comportment. Here, $u(x,\mathrm{t})$ is distributed pressure, $\mathrm{f}(x,\mathrm{t})$ is influx or outflux of fluid, $k(x)$ is vibration in the water body known as transmissibility. $u(x,\mathrm{t})$ is observable at every point, that is, $u^{obs}(x_i,t)$ and always non-negative. Also, we can observe $k(x)$ and $k(x_i)$ at the measurement point $(x_1,\dots,x_n)$. The equation is an inverse problem in mathematics associated with the following equation (2)

$$div(\bar{\bar{T}}\, gradO) + V = \Im \frac{\partial O}{\partial t} \quad (2)$$

Where V is the volume of water, O is the observation from piezometric level, $\bar{\bar{T}}$ is the transmissibility and $\Im$ is storing coefficient. However, the problem is not well-posed as it does not have a unique solution as shown by [2]. Also, the problem is supposed to be a subsurface flow, and the adjacent streamlines are independent of transmissibility of each other. These types of equation are solved using different mathematical methods which provide numerical and sometimes exact solutions. These methods include Adomian decomposition method [3,4], Homotopy perturbation method [5-7], Homotopy analysis Method [8,9], Finite element method [10,11]. A lot of work has been done in this field and mathematics is being used in study of subsurface water flow [12].

In this paper, we will use Homotopy perturbation method (HAM) which converges to exact solution effectively. Only a few iterations are needed for a good enough solution. It is a semi-analytical method to solve nonlinear ODEs and PDEs. The said method employs its concept of homotopy from topology and hence generating a series solution converging to exact solution. Homotopy analysis method was first introduced by Liao Shijun in 1992 [13] and further modified in 1997 [14] in which a convergence controlling parameter was introduced. It is different from other analytical methods in many aspects. It gives a series expansion which does not depends on physical parameters, so it is useful for many types of nonlinear physical problems which are not solvable with standard perturbation method.

2. MATHEMATICAL DESCRIPTION OF HAM

Consider a nonlinear differential equation

$$N[u(x)] = 0 \quad (2.1)$$

Where N is a nonlinear operator. We introduce L, $u(x_0)$ and C as Linear operator, initial guess and convergence controlling parameter respectively to construct a family of equations as follow

$$(1-q)\,\mathrm{L}[\mathrm{U}(\mathrm{x};\mathrm{q}) - \mathrm{u}_0(x)] = c_0 qN[\mathrm{U}(\mathrm{x};\mathrm{q})] \quad (2.2)$$

The above relationship is called zeroth order deformation and the solution of this relationship varies with the embedding parameter $q \in [0,1]$. When $q = 0$ the linear equation pre-decided initial guess reads as

$$\mathrm{L}[\mathrm{U}(\mathrm{x};\mathrm{q}) - \mathrm{u}_0(x)] = 0 \quad (2.3)$$

Expanding the Taylor series of $\mathrm{U}(\mathrm{x};\mathrm{q})$ about $q = 0$, the final form of HAM becomes

$$\mathrm{u}(x) = \mathrm{u}_0(x) \quad \sum_{m=1}^{\infty} \mathrm{u}_m(x) \quad (2.4)$$

3. METHODOLOGY

We consider a particular case of equation (1). For the application of Homotopy perturbation method, the equation should be molded in the following shape [15].

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}, \quad x \in [0,1] \quad (3)$$

With $f(x,t)=0$ and $k(x)=1 \cdot u_t = u_{xx}$

$$u_t = u_{xx} \tag{3.1}$$

with an initial guess as $u(x,0)=e^x$

HAM provides a set of iterations which will be used to evaluate the solution components later.

$$u_{0t} - e^x = 0 : u_0(x,0) = e^x \tag{3.2}$$

$$u_{1t} - u_{0xx} + e^x = 0 : u_1(x,0) = 0 \tag{3.3}$$

$$u_{2t} - u_{1xx} = 0 : u_2(x,0) = 0 \tag{3.4}$$

$$u_{3t} - u_{2xx} = 0 : u_3(x,0) = 0 \tag{3.5}$$

Now, eq (3.2) implies

$$(u_0)_t = e^x \tag{4}$$

Integrating eq (4) with respect to t, we get

$$u_0 = e^x t + c \tag{4.1}$$

Here c is a constant and its value can be calculated as

$$u_0(x,0) = e^x(0) + c \Rightarrow c = e^x \tag{4.2}$$

Putting the value of c in eq (4.1), we get first component of the solution, that is u_0

$$u_0(x,0) = e^x t + e^x = e^x(t+1) \tag{4.3}$$

Similarly, eq (3) implies the following results

$$u_{1t} - u_{0xx} + e^x = 0 \tag{4.4}$$

$$u_{1t} - \lfloor e\ \ (t+1) \rfloor_{xx} + e^x = 0 \tag{4.5}$$

Taking the derivative of the middle term with respect to x twice, we get

$$u_{1t} = e^x(t) \tag{5}$$

Integrating with respect to t, we get u_1

$$u_1 = \frac{e^x t^2}{2} \tag{5.1}$$

Eq (3.4) provides the following results,

$$u_{2t} - u_{1xx} = 0 \tag{5.2}$$

Putting the value of u_1, eq (5.3) becomes

$$u_{2t} = \left[\frac{e^x t^2}{2}\right]_{xx} \tag{5.3}$$

and taking derivative twice with respect to x

$$u_{2t} = \frac{e^x t^2}{2} \tag{5.4}$$

Integrating with respect to t we get

$$u_2 = \frac{e^x t^3}{6} \tag{5.5}$$

Following the same procedure, eq (3.5) produces the component u_3 as

$$u_3 = \frac{e^x t^4}{24} \tag{6}$$

Adding all the components will give a series solution that approaches a convergent solution [4].

$$u = u_0 + u_1 + u_2 + u_3 + \ldots \tag{6.1}$$

Putting the values of all components using eq (4.3), (5.1), (5.5) and (6) produces

$$u = e^x(t+1) + \frac{e^x t}{2} + \frac{e^x t^3}{6} + \frac{e^x t}{24} + \ldots \tag{6.2}$$

Simplification of above equation gives

$$u = e^x\left[1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \ldots\right] \tag{6.3}$$

$$u = e^x \cdot e^t \tag{6.4}$$

$$u = e^{x+t} \tag{6.5}$$

The eq(6.5) gives the pressure profile of water-bearing explained by equation (1). The results show that the pressure generated in water reservoir is dependent on distance *x* covered by water in time *t*. The pressure grows exponentially in the direction of flow at a specific time *t*.

4. RESULTS

The absolute error is defined as $error = |u_{Exact} - u_{HAM}|$. Moreover, it is evident from the results that the absolute error can be made lesser by the addition of successive terms using iterational formula. The final results show the distributed pressure $u(x,t)$ profile of fluid under discussion. Here t is the time needed to cover the distance x which determines the distributed pressure profile of fluid. The graphical results shows the consistency and accuracy of HAM Approximated solution.

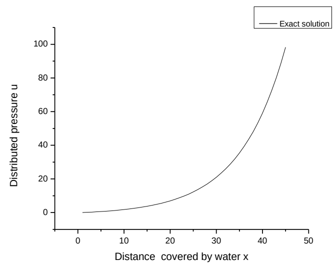

Figure 1: Shows The graphical results of exact solution.

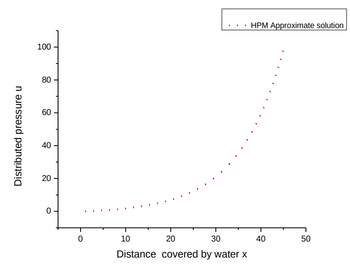

Figure 2: Shows the approximate solution of problem achieved by HAM.

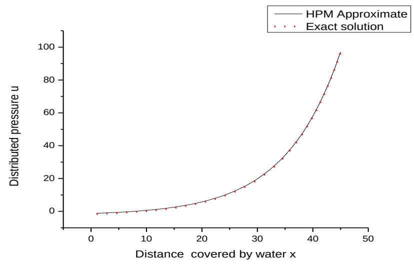

Figure 3: Shows the comparison between approximate and exact solution.

5. CONCLUSION

A partial differential equation from hydrology is solved with HAM. It is concluded that all components of the solution are obtained recurrently without discretization of time and space. It was indicated that to verify the numerical accuracy of proposed methodology, final results are evaluated numerically with arbitrary values of variables. Furthermore, it can be calculated using simple mathematics that the error between exact and numerical solution is getting minimum with each iteration of solution using Homotopy analysis method. The comparison between approximate and exact solution show the reliability of HAM approximated solution.

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